

Relation between hyperbolic Nizhnik-Novikov-Veselov equation and stationary Davey-Stewartson II equation

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Abstract

A Lax system in three variables is presented, two equations of which form the Lax pair of the stationary Davey-Stewartson II equation. With certain nonlinear constraints, the full integrability condition of this Lax system contains the hyperbolic Nizhnik-Novikov-Veselov equation and its standard Lax pair. The Darboux transformation for the Davey-Stewartson II equation is used to solve the hyperbolic Nizhnik-Novikov-Veselov equation. Using Darboux transformation, global n -soliton solutions are obtained. It is proved that each n -soliton solution approaches zero uniformly and exponentially at spatial infinity and is asymptotic to n^2 lumps of peaks at temporal infinity.

1 Introduction

The Nizhnik-Novikov-Veselov (NNV) equation [16, 17, 19] is an important 2+1 dimensional integrable equation which is a natural generalization of the KdV equation to 2+1 dimensions. It is useful in both mechanics and differential geometry [11, 12]. The NNV equation has been solved by various methods such as inverse scattering [2], bilinear method [18], bilinear Bäcklund transformation [8], binary Darboux transformation [14] and so on [1, 6, 7, 9, 13, 15]. However, one can not construct the usual Darboux transformation (without integration) because the principal part of the first equation of its Lax pair is two dimensional wave operator or Laplace operator.

Starting from the idea of nonlinearization [3], many high dimensional integrable systems were reduced to lower dimensional ones so that interesting solutions like soliton solutions and quasi-periodic solutions can be obtained from lower dimensional systems. Especially, the KP equation [4, 10], the DSI equation and the 2 + 1 dimensional N -wave equation [21] were related to some 1+1 dimensional AKNS systems. Following this idea, in this paper, we present a Lax system of three variables, two equations of which form the Lax pair of the

stationary Davey-Stewartson II (DS II) equation. With the nonlinear constraints (14), the full integrability condition of this Lax system contains the hyperbolic NNV equation and its standard Lax pair.

The DSII equation has a Darboux transformation without integration. With the relations given by (14), the Darboux transformation for DSII equation is used to solve the hyperbolic NNV equation. This Darboux transformation without integration is more suitable for symbolic calculation than the known binary Darboux transformation.

It is well known that DSI equation has solutions approaching zero exponentially at spatial infinity, but DSII equation has not. However, we get soliton solution u of the hyperbolic NNV equation from that of the stationary DSII equation so that u approaches zero exponentially at spatial infinity. This is possible because the solution u of the hyperbolic NNV equation is given by $i(g - \bar{g})$ as in (14), not f , the solution of the stationary DSII equation. These soliton solutions are different from the known one derived by binary Darboux transformation or bilinear method etc. and the behavior of the solutions is more complicated.

In Section 2, after reviewing the hyperbolic NNV equation and the stationary DSII equation together with their standard Lax pairs, a new Lax system (10) is presented in which an extra equation is added to the standard Lax pair of the stationary DSII equation. With the nonlinear constraints (14), the integrability condition of this Lax system includes both the hyperbolic NNV equation and its standard Lax pair. The Darboux transformation for the new Lax system is given in Section 3 and the general expression of multi-soliton solutions is presented in Section 4. In Section 5, the explicit expressions and behavior of single-soliton solutions are discussed. In Section 6, it is proved that each n -soliton solution approaches zero uniformly and exponentially at spatial infinity. In Section 7, it is proved that each n -soliton solution is asymptotic to n^2 lumps of peaks at temporal infinity. Finally, some linear algebraic lemmas are presented in the Appendix.

2 Hyperbolic Nizhnik-Novikov-Veselov equation and Davey-Stewartson II equation

The hyperbolic NNV equation is

$$\begin{aligned} u_t &= u_{\xi\xi\xi} + u_{\eta\eta\eta} + 3(uv)_\xi + 3(uw)_\eta, \\ v_\eta &= u_\xi, \quad w_\xi = u_\eta, \end{aligned} \tag{1}$$

which has a Lax pair

$$\begin{aligned} f_{\xi\eta} + uf &= 0, \\ f_t &= f_{\xi\xi\xi} + f_{\eta\eta\eta} + 3vf_\xi + 3wf_\eta. \end{aligned} \tag{2}$$

By taking the new coordinates $x = \xi - \eta$, $y = \xi + \eta$, the hyperbolic NNV equation (1) becomes

$$\begin{aligned} u_t &= 2u_{yyy} + 6u_{xxy} + 3(u(v+w))_y + 3(u(v-w))_x, \\ (\partial_y - \partial_x)v &= (\partial_y + \partial_x)u, \quad (\partial_y + \partial_x)w = (\partial_y - \partial_x)u, \end{aligned} \tag{3}$$

and the Lax pair (2) becomes

$$\begin{aligned} f_{yy} - f_{xx} + uf &= 0, \\ f_t &= 2f_{yyy} + 6u_{xy} + 3(v+w)f_y + 3(v-w)f_x. \end{aligned} \quad (4)$$

On the other hand, the DSII equation is

$$\begin{aligned} -if_\tau &= f_{xx} - f_{yy} - i(g - \bar{g})f, \\ (\partial_y - i\partial_x)g &= (\partial_x - i\partial_y)(|f|^2), \end{aligned} \quad (5)$$

which has a Lax pair

$$\begin{aligned} \Psi_y &= iJ\Psi_x + P\Psi, \\ \Psi_\tau &= 2iJ\Psi_{xx} + 2P\Psi_x + Q\Psi \end{aligned} \quad (6)$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} & f \\ -\bar{f} & \end{pmatrix}, \quad Q = \begin{pmatrix} g & f_x - if_y \\ -\bar{f}_x - i\bar{f}_y & \bar{g} \end{pmatrix}. \quad (7)$$

If (f, g) is independent of τ , (5) becomes the stationary DSII equation

$$\begin{aligned} f_{xx} - f_{yy} - i(g - \bar{g})f &= 0, \\ (\partial_y - i\partial_x)g &= (\partial_x - i\partial_y)(|f|^2). \end{aligned} \quad (8)$$

Taking $\Psi(x, y, \tau) = \Phi(x, y)e^{2i\lambda^2\tau}$ in (6), we get the Lax pair for (8) as

$$\begin{aligned} \Phi_y &= iJ\Phi_x + P\Phi, \\ 2i\lambda^2\Phi &= 2iJ\Phi_{xx} + 2P\Phi_x + Q\Phi \end{aligned} \quad (9)$$

The first equation of (4) and the first equation of (8) are similar, and the second equation of (4) is of order 3. Hence we introduce an extra equation to the Lax pair (9) so that the whole system becomes

$$\begin{aligned} \Phi_y &= M(\partial)\Phi \equiv iJ\Phi_x + P\Phi, \\ 2i\lambda^2\Phi &= L(\partial)\Phi \equiv 2iJ\Phi_{xx} + 2P\Phi_x + Q\Phi, \\ \Phi_t &= N(\partial)\Phi \equiv 16iJ\Phi_{xxx} + 16P\Phi_{xx} + R\Phi_x + S\Phi \end{aligned} \quad (10)$$

where J, P, Q are given by (7),

$$\begin{aligned} R &= 4 \begin{pmatrix} 3g + i|f|^2 & 4f_x - 2if_y \\ -4\bar{f}_x - 2i\bar{f}_y & 3\bar{g} - i|f|^2 \end{pmatrix}, \\ S &= 2 \begin{pmatrix} 3g_x + 2i\bar{f}f_x + \bar{f}f_y - f\bar{f}_y & 6f_{xx} - 2if_{xy} - i(g - \bar{g})f + 2|f|^2f \\ -6\bar{f}_{xx} - 2i\bar{f}_{xy} + i(g - \bar{g})\bar{f} - 2|f|^2\bar{f} & 3\bar{g}_x - 2i\bar{f}\bar{f}_x + \bar{f}\bar{f}_y - \bar{f}f_y \end{pmatrix}, \end{aligned} \quad (11)$$

and $L(\partial)$, $M(\partial)$ and $N(\partial)$ refer to differential operators with respect to x whose coefficients are 2×2 matrices, $\partial = \partial_x$.

The integrability conditions of (10) include the following equations:

$$\begin{aligned} f_{yy} - f_{xx} + uf &= 0, \\ f_t &= 2f_{yyy} + 6f_{xxy} + 3(v+w)f_y + 3(v-w)f_x, \end{aligned} \quad (12)$$

$$\begin{aligned} (\partial_y - i\partial_x)g &= (\partial_x - i\partial_y)(|f|^2), \\ \frac{i}{2}g_t &= -2g_{xxx} + 2\bar{f}f_{xxy} + 2f\bar{f}_{xxy} + 4i\bar{f}f_{xxx} + 4if\bar{f}_{xxx} \\ &\quad + 2(\bar{f}_x - i\bar{f}_y)f_{xy} + 2(f_x - if_y)\bar{f}_{xy} + 2(i\bar{f}_x + 2\bar{f}_y)f_{xx} + 2(if_x + 2f_y)\bar{f}_{xx} \\ &\quad + (2|f|^2 - i(g - \bar{g}))(|f|^2)_y + (6i|f|^2 + (g - \bar{g}))(|f|^2)_x - 2|f|^2\bar{g}_x + 6ig\bar{g}_x, \end{aligned} \quad (13)$$

where

$$u = i(g - \bar{g}), \quad v = 2|f|^2 + (g + \bar{g}), \quad w = 2|f|^2 - (g + \bar{g}). \quad (14)$$

Note that (12) is exactly the same as the original Lax pair (4) of the hyperbolic NNV equation. By direct calculation, we know that (u, v, w) satisfies the hyperbolic NNV equation (3) provided that f and g satisfy (12)–(14). Therefore, explicit solutions of the hyperbolic NNV equation can be obtained from those of (12)–(14).

Clearly, the solutions of (12)–(14) are only part of those of the hyperbolic NNV equation. However, they include some interesting ones which will be shown in the rest of this paper.

3 Darboux transformation

The binary Darboux transformation for the hyperbolic NNV equation is well-known [14]. Integrations are needed in constructing explicit solutions. However, for DSII equation, usual Darboux transformation without integration is known. This Darboux transformation is simpler than the binary Darboux transformation for the hyperbolic NNV equation, and can be easily used to the stationary DSII equation so that explicit solutions of the hyperbolic NNV equation can be constructed.

Note that the coefficients of $L(\partial)$, $M(\partial)$, $N(\partial)$ satisfy

$$iJ, P, Q, R, S \in \Sigma \quad (15)$$

where

$$\Sigma = \{A \text{ is a } 2 \times 2 \text{ matrix} \mid KAK^{-1} = \bar{A}\} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbf{C} \right\}, \quad (16)$$

$K = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. That is, $L(\partial)$, $M(\partial)$, $N(\partial)$ satisfy

$$KL(\partial)K^{-1} = \bar{L}(\partial), \quad KM(\partial)K^{-1} = \bar{M}(\partial), \quad KN(\partial)K^{-1} = \bar{N}(\partial). \quad (17)$$

Hence, if $\Phi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ is a solution of (10) with $\lambda = \lambda_0$, then $K\bar{\Phi} = \begin{pmatrix} -\bar{\eta} \\ \bar{\xi} \end{pmatrix}$ is a solution of (10) with $\lambda = \pm i\bar{\lambda}_0$.

The Darboux transformation of arbitrary order is constructed as follows [5, 20]. Suppose

$$G(\partial) = \partial^n + G_1(x, y, t)\partial^{n-1} + \cdots + G_n(x, y, t) \quad (18)$$

is a Darboux operator for (10), i.e., there exist $L'(\partial)$, $M'(\partial)$, $N'(\partial)$ which have the same form as $L(\partial)$, $M(\partial)$, $N(\partial)$ with f and g replaced by certain f' and g' , such that $\Phi' = G(\partial)\Phi$ satisfies

$$\lambda\Phi' = L'(\partial)\Phi', \quad \Phi'_y = M'(\partial)\Phi', \quad \Phi'_t = N'(\partial)\Phi'. \quad (19)$$

If so, $G(\partial)$ satisfies

$$\begin{aligned} L'(\partial)G(\partial) &= G(\partial)L(\partial), \\ M'(\partial)G(\partial) &= G(\partial)M(\partial) + G_y(\partial), \\ N'(\partial)G(\partial) &= G(\partial)N(\partial) + G_t(\partial). \end{aligned} \quad (20)$$

Since $L(\partial)$, $M(\partial)$, $N(\partial)$ satisfy the relations (17), and $L'(\partial)$, $M'(\partial)$, $N'(\partial)$ satisfy the similar relations

$$KL'(\partial)K^{-1} = \bar{L}'(\partial), \quad KM'(\partial)K^{-1} = \bar{M}'(\partial), \quad KN'(\partial)K^{-1} = \bar{N}'(\partial), \quad (21)$$

we want that $G(\partial)$ satisfies $KG(\partial)K^{-1} = \bar{G}(\partial)$. Write

$$G_j = \begin{pmatrix} a_j & b_j \\ -\bar{b}_j & \bar{a}_j \end{pmatrix}. \quad (22)$$

Denote $L'(\partial) = 2iJ\partial^2 + 2P'\partial + Q'$, then the first equation of (20) leads to

$$\begin{aligned} (2iJ\partial^2 + 2P'\partial + Q')(\partial^n + G_1\partial^{n-1} + \cdots + G_n) \\ = (\partial^n + G_1\partial^{n-1} + \cdots + G_n)(2iJ\partial^2 + 2P\partial + Q), \end{aligned} \quad (23)$$

in which the coefficients of ∂^{n+1} and ∂^n give

$$\begin{aligned} P' &= P - i[J, G_1], \\ Q' &= Q - 2i[J, G_2] - 2[P, G_1] + 2i[J, G_1]G_1 + 2nP_x - 4iJG_{1,x}. \end{aligned} \quad (24)$$

Hence, after the action of Darboux transformation,

$$\begin{aligned} f' &= f - 2ib_1, \\ g' &= g - 4ia_{1,x} - 2(\bar{f}b_1 - f\bar{b}_1) - 4i|b_1|^2, \\ u' &= u + 8|b_1|^2 - 4i(\bar{f}b_1 - f\bar{b}_1) + 4(a_1 + \bar{a}_1)_x, \\ v' &= v + 8|b_1|^2 - 4i(\bar{f}b_1 - f\bar{b}_1) - 4i(a_1 - \bar{a}_1)_x, \\ w' &= w + 8|b_1|^2 - 4i(\bar{f}b_1 - f\bar{b}_1) + 4i(a_1 - \bar{a}_1)_x. \end{aligned} \quad (25)$$

Now take n distinct complex numbers $\lambda_1, \dots, \lambda_n$ with $\lambda_j = \mu_j + i\nu_j$ (μ_j 's and ν_j 's are real). Let $\Phi_j = \begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix}$ be a column solution of (10) with $\lambda = \lambda_j$, then $\Phi_{n+j} \equiv K\bar{\Phi}_j = \begin{pmatrix} -\bar{\eta}_j \\ \bar{\xi}_j \end{pmatrix}$ is a solution of (10) with $\lambda = \pm i\bar{\lambda}_j$ ($j = 1 \dots, n$). The Darboux transformation is determined by the system of linear algebraic equations

$$G(\partial)\Phi_j = 0 \quad (j = 1, \dots, 2n) \quad (26)$$

if it has a unique solution [20].

Denote

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad (27)$$

then (26) becomes

$$T \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} \partial^n \xi \\ -\partial^n \bar{\eta} \end{pmatrix} \quad (28)$$

where

$$T = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \quad (29)$$

$$A = \begin{pmatrix} \partial^{n-1} \xi & \dots & \xi \end{pmatrix}, \quad B = \begin{pmatrix} \partial^{n-1} \eta & \dots & \eta \end{pmatrix}. \quad (30)$$

(26) has a unique solution if and only if $\det T \neq 0$.

4 Expression of soliton solutions

For zero seed solution $u = v = w = f = g = 0$, (10) becomes

$$\Phi_{xx} = \lambda^2 J \Phi, \quad \Phi_y = iJ \Phi_x, \quad \Phi_t = 16iJ \Phi_{xxx} \quad (31)$$

with $\Phi = (\xi, \eta)^T$. Hence take

$$\xi_j = \kappa_j^{(1)} (e^{\rho_j^{(1)} + i\sigma_j^{(1)}} + e^{-\rho_j^{(1)} - i\sigma_j^{(1)}}), \quad \eta_j = \kappa_j^{(2)} (e^{\rho_j^{(2)} + i\sigma_j^{(2)}} + e^{-\rho_j^{(2)} - i\sigma_j^{(2)}}) \quad (32)$$

where

$$\begin{aligned} \rho_j^{(1)} &= \text{Re}(\lambda_j x + i\lambda_j y + 16i\lambda_j^3 t) + \rho_{j0}^{(1)} = \mu_j x - \nu_j y + 16(\nu_j^3 - 3\mu_j^2 \nu_j)t + \rho_{j0}^{(1)}, \\ \rho_j^{(2)} &= \text{Re}(i\lambda_j x + \lambda_j y - 16\lambda_j^3 t) + \rho_{j0}^{(2)} = -\nu_j x + \mu_j y - 16(\mu_j^3 - 3\mu_j \nu_j^2)t + \rho_{j0}^{(2)}, \\ \sigma_j^{(1)} &= \text{Im}(\lambda_j x + i\lambda_j y + 16i\lambda_j^3 t) + \sigma_{j0}^{(1)} = \nu_j x + \mu_j y + 16(\mu_j^3 - 3\mu_j \nu_j^2)t + \sigma_{j0}^{(1)}, \\ \sigma_j^{(2)} &= \text{Im}(i\lambda_j x + \lambda_j y - 16\lambda_j^3 t) + \sigma_{j0}^{(2)} = \mu_j x + \nu_j y + 16(\nu_j^3 - 3\mu_j^2 \nu_j)t + \sigma_{j0}^{(2)}, \end{aligned} \quad (33)$$

$\kappa_j^{(1)}, \kappa_j^{(2)}$ are non-zero constants, $\rho_{j0}^{(1)}, \rho_{j0}^{(2)}, \sigma_{j0}^{(1)}, \sigma_{j0}^{(2)}$ are real constants. By solving a_j 's and b_j 's from (28), the Darboux transformation (25) gives the n -soliton solution

$$\begin{aligned} f &= -2ib_1, \quad g = -4ia_{1,x} - 4i|b_1|^2, \\ u &= 8|b_1|^2 + 4(a_1 + \bar{a}_1)_x, \quad v = 8|b_1|^2 - 4i(a_1 - \bar{a}_1)_x, \quad w = 8|b_1|^2 + 4i(a_1 - \bar{a}_1)_x. \end{aligned} \quad (34)$$

Hereafter we omit the primes on f, g, u, v, w for those obtained by the action of Darboux transformation.

Let $K_n = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$. Denote

$$\zeta = \begin{pmatrix} \xi \\ -\bar{\eta} \end{pmatrix}, \quad R_{j\dots k} = \begin{pmatrix} \partial^j \zeta & \partial^{j-1} \zeta, \dots, \partial^k \zeta \end{pmatrix} \quad (35)$$

for $j \geq k$, then

$$T = \begin{pmatrix} R_{n-1\dots 0} & K_n \bar{R}_{n-1\dots 0} \end{pmatrix}. \quad (36)$$

Let

$$\Pi = \begin{pmatrix} \partial^n \zeta & \partial^{n-1} \zeta & \partial^{n-2} \zeta & R_{n-3\dots 0} & K_n \bar{R}_{n-1\dots 0} & 0 & 0 \\ \partial^{n+1} \zeta & \partial^n \zeta & \partial^{n-1} \zeta & 0 & 0 & R_{n-2\dots 0} & K_n \bar{R}_{n-1\dots 0} \end{pmatrix}. \quad (37)$$

Theorem 1 *When $\det T \neq 0$, the multi-soliton solution u of the hyperbolic NNV equation given by (34) can be written as*

$$u = -8 \frac{\operatorname{Re} \det \Pi}{(\det T)^2}. \quad (38)$$

Proof. Solved from (28) by Cramer rule,

$$a_1 = -(\det T)^{-1} \begin{vmatrix} \partial^n \zeta & R_{n-2\dots 0} & K_n \bar{R}_{n-1\dots 0} \end{vmatrix}, \quad (39)$$

$$b_1 = -(\det T)^{-1} \begin{vmatrix} R_{n-1\dots 0} & \partial^n \zeta & K_n \bar{R}_{n-2\dots 0} \end{vmatrix}. \quad (40)$$

$$\begin{aligned} & a_1 + \bar{a}_1 \\ &= -(\det T)^{-1} \left(\begin{vmatrix} \partial^n \zeta & R_{n-2\dots 0} & K_n \bar{R}_{n-1\dots 0} \end{vmatrix} + \overline{\begin{vmatrix} \partial^n \zeta & R_{n-2\dots 0} & K_n \bar{R}_{n-1\dots 0} \end{vmatrix}} \right) \\ &= -(\det T)^{-1} \left(\begin{vmatrix} \partial^n \zeta & R_{n-2\dots 0} & K_n \bar{R}_{n-1\dots 0} \end{vmatrix} + \begin{vmatrix} R_{n-1\dots 0} & K_n \partial^n \bar{\zeta} & K_n \bar{R}_{n-2\dots 0} \end{vmatrix} \right) \\ &= -(\det T)^{-1} (\det T)_x = -\operatorname{tr}(T^{-1} T_x), \end{aligned} \quad (41)$$

$$a_{1,x} + \bar{a}_{1,x} = -\operatorname{tr}(T^{-1} T_{xx}) + \operatorname{tr}((T^{-1} T_x)^2). \quad (42)$$

Denote $\tilde{I}_k = \begin{pmatrix} I_{k \times k} \\ 0_{(n-k) \times k} \end{pmatrix}$. Let $h = \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}$ be the solution of $Th = -\partial^{n+1}\zeta$ where $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_n)^T$, $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_n)^T$, then

$$\begin{aligned} a_{1,x} + \bar{a}_{1,x} &= -\text{tr} \begin{pmatrix} -\tilde{a} & -a & \tilde{I}_{n-2} & \bar{\tilde{b}} & \bar{b} & 0 \\ -\tilde{b} & -b & 0 & -\tilde{a} & -\bar{a} & \tilde{I}_{n-2} \end{pmatrix} \\ &\quad + \text{tr} \begin{pmatrix} -a & \tilde{I}_{n-1} & \bar{b} & 0 \\ -b & 0 & -\bar{a} & \tilde{I}_{n-1} \end{pmatrix}^2 \\ &= a_1^2 + \bar{a}_1^2 + \tilde{a}_1 + \bar{\tilde{a}}_1 - a_2 - \bar{a}_2 - 2|b_1|^2. \end{aligned} \quad (43)$$

According to (34),

$$u = 8\text{Re}(a_1^2 + \tilde{a}_1 - a_2). \quad (44)$$

Let $d = -(\det T)^2(a_1^2 + \tilde{a}_1 - a_2)$, then by Cramer rule,

$$\begin{aligned} d &= - \begin{vmatrix} \partial^n \zeta & R_{n-2 \dots 0} & K_n \bar{R}_{n-1 \dots 0} \\ \partial^n \zeta & \partial^{n-1} \zeta & R_{n-3 \dots 0} & K_n \bar{R}_{n-1 \dots 0} \end{vmatrix}^2 + \det T \begin{vmatrix} \partial^{n+1} \zeta & R_{n-2 \dots 0} & K_n \bar{R}_{n-1 \dots 0} \\ \partial^n \zeta & \partial^{n-1} \zeta & R_{n-3 \dots 0} & K_n \bar{R}_{n-1 \dots 0} \end{vmatrix}. \end{aligned} \quad (45)$$

Using Laplace expansion of $\det \Pi$, $d = \det \Pi$. Hence

$$u = -8 \frac{\text{Re } d}{(\det T)^2} = -8 \frac{\text{Re } \det \Pi}{(\det T)^2}. \quad (46)$$

The theorem is proved.

Remark 1 According to Lemma 2 of A, $\det T \geq 0$ holds everywhere. However, $\det T > 0$ may not hold everywhere when the parameters $\rho_{j0}^{(k)}$ and $\sigma_{j0}^{(k)}$ take some special values, as will shown in the next section for single soliton solution. On the other hand, $\det T > 0$ holds everywhere in generic case, which will be shown here.

The Darboux operator $G(\partial)$ of order n can be constructed by composing n Darboux operators of order one as follows. For given $\lambda_1, \dots, \lambda_n$ and Φ_1, \dots, Φ_n as above, let $H_j = (\Phi_j, \Phi_{n+j})$ $j(= 1, \dots, n)$. If $\det H_1 \neq 0$, then $\Delta_1(\partial) = \partial - H_{1,x} H_1^{-1}$ is a Darboux operator of order one. It transforms (u, v, w, f, g) to $(u^{(1)}, v^{(1)}, w^{(1)}, f^{(1)}, g^{(1)})$ and transforms H_j to $H_j^{(1)} = \Delta_1(\partial) H_j = H_{j,x} - H_{1,x} H_1^{-1} H_j$ ($j = 2, 3, \dots, n$). Again, if $\det H_2^{(1)} \neq 0$, then $\Delta_2(\partial) = \partial - H_{2,x}^{(1)} (H_2^{(1)})^{-1}$ is a Darboux operator of order one for the Lax pair with $(u^{(1)}, v^{(1)}, w^{(1)}, f^{(1)}, g^{(1)})$. It transforms $(u^{(1)}, v^{(1)}, w^{(1)}, f^{(1)}, g^{(1)})$ to $(u^{(2)}, v^{(2)}, w^{(2)}, f^{(2)}, g^{(2)})$ and transforms $H_j^{(1)}$ to $H_j^{(2)} = \Delta_2(\partial) H_j^{(1)} = H_{j,x}^{(1)} - H_{2,x}^{(1)} (H_2^{(1)})^{-1} H_j^{(1)}$ ($j = 3, 4, \dots, n$). Continuing this process, we get $H_j^{(k)}$ ($k = 1, \dots, n-1$; $j = k+1, \dots, n$) and $\Delta_j(\partial)$ ($j = 1, \dots, n$). According to [20],

$$\begin{aligned} G(\partial) &= \Delta_n(\partial) \Delta_{n-1}(\partial) \cdots \Delta_1(\partial), \\ \det T &= \det(H_n^{(n-1)}) \det(H_{n-1}^{(n-2)}) \cdots \det(H_2^{(1)}) \det(H_1). \end{aligned} \quad (47)$$

Hence $\det T \neq 0$ if all $\det H_j^{(j-1)} \neq 0$.

Suppose $H_j^{(j-1)} = \begin{pmatrix} \xi_j^{(j-1)} & -\bar{\eta}_j^{(j-1)} \\ \eta_j^{(j-1)} & \bar{\xi}_j^{(j-1)} \end{pmatrix}$, then $\det H_j^{(j-1)} = |\xi_j^{(j-1)}|^2 + |\eta_j^{(j-1)}|^2 = 0$ if and only if $\xi_j^{(j-1)} = 0$ and $\eta_j^{(j-1)} = 0$ hold simultaneously. For fixed j , this gives a system of four real equations

$$\operatorname{Re} \xi_j^{(j-1)} = 0, \quad \operatorname{Im} \xi_j^{(j-1)} = 0, \quad \operatorname{Re} \eta_j^{(j-1)} = 0, \quad \operatorname{Im} \eta_j^{(j-1)} = 0 \quad (48)$$

for three real variables x, y, t . It has no solution unless the parameters $\rho_{j0}^{(k)}$ and $\sigma_{j0}^{(k)}$ ($j = 1, \dots, n; k = 1, 2$) take special values. This shows that $\det T > 0$ holds everywhere for generic $\rho_{j0}^{(k)}$ and $\sigma_{j0}^{(k)}$. Therefore, the multi-soliton solution u is global for generic $\rho_{j0}^{(k)}$ and $\sigma_{j0}^{(k)}$.

5 Single soliton solution

By taking $n = 1$, the single soliton can be obtained as

$$u = \frac{16A}{B^2} \quad (49)$$

where

$$\begin{aligned} B &= (\kappa_1^{(1)})^2 \cosh(2\rho_1^{(1)}) + (\kappa_1^{(2)})^2 \cosh(2\rho_1^{(2)}) + (\kappa_1^{(1)})^2 \cos(2\sigma_1^{(1)}) + (\kappa_1^{(2)})^2 \cos(2\sigma_1^{(2)}), \\ A &= -(\mu_1^2 - \nu_1^2)(\kappa_1^{(1)})^4 \cosh(2\rho_1^{(1)}) \cos(2\sigma_1^{(1)}) - 2\mu_1\nu_1(\kappa_1^{(1)})^4 \sinh(2\rho_1^{(1)}) \sin(2\sigma_1^{(1)}) \\ &\quad - (\mu_1^2 + \nu_1^2)(\kappa_1^{(1)})^2(\kappa_1^{(2)})^2 \sinh(2\rho_1^{(1)}) \sin(2\sigma_1^{(2)}) \\ &\quad + (\mu_1^2 - \nu_1^2)(\kappa_1^{(2)})^4 \cosh(2\rho_1^{(2)}) \cos(2\sigma_1^{(2)}) + 2\mu_1\nu_1(\kappa_1^{(2)})^4 \sinh(2\rho_1^{(2)}) \sin(2\sigma_1^{(2)}) \\ &\quad + (\mu_1^2 + \nu_1^2)(\kappa_1^{(1)})^2(\kappa_1^{(2)})^2 \sinh(2\rho_1^{(2)}) \sin(2\sigma_1^{(1)}) \\ &\quad + (\mu_1^2 - \nu_1^2)((\kappa_1^{(2)})^4 - (\kappa_1^{(1)})^4). \end{aligned} \quad (50)$$

The solution is singular if $B = 0$, i.e. $|\xi_1|^2 + |\eta_1|^2 = 0$. This is equivalent to $\rho_1^{(1)} = \rho_1^{(2)} = 0$, $2\sigma_1^{(1)} = j\pi + \pi/2$, $2\sigma_1^{(2)} = k\pi + \pi/2$ for certain integers j and k . In contrast, the solution is global if and only if $|\xi_1|^2 + |\eta_1|^2 \neq 0$ everywhere, i.e. the parameters satisfy

$$\mu_1(\rho_{10}^{(1)} - \sigma_{10}^{(2)} + k\pi + \pi/2) + \nu_1(\rho_{10}^{(2)} + \sigma_{10}^{(1)} - j\pi - \pi/2) \neq 0. \quad (51)$$

We always suppose (51) is satisfied, which is equivalent to $\det T \neq 0$.

When $\mu_1^2 \neq \nu_1^2$, the solution u approaches zero exponentially at spatial infinity, and the peaks appear when neither $\rho_1^{(1)}$ nor $\rho_1^{(2)}$ is large. Hence the center of the lump of peaks locates near $\rho_1^{(1)} = 0$ and $\rho_1^{(2)} = 0$, i.e.,

$$x = 64\mu_1\nu_1 t - \frac{\mu_1\rho_{10}^{(1)} + \nu_1\rho_{10}^{(2)}}{\mu_1^2 - \nu_1^2}, \quad y = 16(\mu_1^2 + \nu_1^2)t - \frac{\nu_1\rho_{10}^{(1)} + \mu_1\rho_{10}^{(2)}}{\mu_1^2 - \nu_1^2}. \quad (52)$$

Figure 1: Single soliton solution u : $\lambda_1 = 2 + 0.5i$, $\kappa_1^{(1)} = 1$, $\kappa_1^{(2)} = 1.2$, $\rho_{10}^{(1)} = 0$, $\rho_{10}^{(2)} = 1$, $\sigma_{10}^{(1)} = \sigma_{10}^{(2)} = 0$, $t = 1$.

The solutions are shown in Figure 1 and Figure 2 for different parameters. The figure of the solution contains a lump of peaks rather than a single peak, and the shape depends on the angle $\arctan \frac{\mu_1^2 - \nu_1^2}{2\mu_1\nu_1}$ between the straight lines $\rho_1^{(1)} = 0$ and $\rho_1^{(2)} = 0$. Nevertheless, we still call it single soliton solution because it is generated from the zero solution by Darboux transformation, and the peaks in the solution never separate.

Note that although u is localized, v and w are not.

If $\nu_1 = \mu_1 \neq 0$, the solution is invariant when (x, y) is changed to $(x + \frac{m\pi}{2\mu_1}, y + \frac{m\pi}{2\mu_1})$ for any integer m . Hence the solution is periodic. Moreover, $\rho_1^{(1)} + \rho_1^{(2)} = \rho_{10}^{(1)} + \rho_{10}^{(2)}$. The peaks appear when neither $\rho_1^{(1)}$ nor $\rho_1^{(2)}$ is large. Hence the peaks lie near the straight line $x - y - 32\mu_1^2 t + \frac{\rho_{10}^{(1)} - \rho_{10}^{(2)}}{2\mu_1} = 0$. The solution is shown in Figure 3.

Similarly, the solution is also periodic if $\nu_1 = -\mu_1 \neq 0$.

6 Localization of the solutions

In this section, we will prove that the multi-soliton solutions approach zero uniformly and exponentially at spatial infinity. In order to get global solutions, we always suppose $\det T \neq 0$ everywhere, which is true for generic parameters $\rho_{j0}^{(k)}$ and $\sigma_{j0}^{(k)}$ ($j = 1, \dots, n$; $k = 1, 2$).

Note that the solution of (28) is invariant if both ξ_j and η_j (for fixed j) are multiplied by a common function. Let

$$\omega_j = \begin{cases} \xi_j & \text{if } |\xi_j| \geq |\eta_j| \\ \eta_j & \text{if } |\xi_j| < |\eta_j| \end{cases}, \quad (53)$$

$$\overset{\circ}{T} = \text{diag}(\omega_1, \dots, \omega_n, \bar{\omega}_1, \dots, \bar{\omega}_n). \quad (54)$$

Figure 2: Single soliton solution u : $\lambda_1 = 1.1 + 0.9i$, $\kappa_1^{(1)} = 1$, $\kappa_1^{(2)} = 1.2$, $\rho_{10}^{(1)} = 0$, $\rho_{10}^{(2)} = 1$, $\sigma_{10}^{(1)} = \sigma_{10}^{(2)} = 0$, $t = 1$.

Figure 3: Periodic solution u : $\lambda_1 = 1 + i$, $\kappa_1^{(1)} = 1$, $\kappa_1^{(2)} = 1.2$, $\rho_{10}^{(1)} = 0$, $\rho_{10}^{(2)} = 1$, $\sigma_{10}^{(1)} = \sigma_{10}^{(2)} = 0$, $t = 1$.

Let $\tilde{T} = \tilde{T}^{-1}T$, then the norm of each entry of \tilde{T} cannot exceed 1. Although \tilde{T} is not continuous, $|\det T|$ is continuous.

Let $x = r \cos \theta$, $y = r \sin \theta$. Since $\rho_j^{(1)}$'s and $\rho_j^{(2)}$'s depend on x and y linearly, we can write, for $k = 1, 2$,

$$\rho_j^{(k)}(r \cos \theta, r \sin \theta, t) = \varepsilon_j^{(k)}(\theta) \alpha_j^{(k)}(\theta) r + \beta_j^{(k)}, \quad (55)$$

where $\varepsilon_j^{(k)}(\theta) = \pm 1$ ($j = 1, \dots, n$) so that $\alpha_j^{(k)}(\theta) \geq 0$. Here the variable t is omitted in $\alpha_j^{(k)}(\theta)$, $\beta_j^{(k)}$ and $\varepsilon_j^{(k)}(\theta)$.

Clearly $\alpha_j^{(k)}$'s are continuous functions. Note also that $\varepsilon_j^{(k)}(\theta)$ is not well-defined when $\alpha_j^{(k)}(\theta) = 0$.

Theorem 2 Suppose $\lambda_1, \dots, \lambda_n$ are distinct non-zero complex numbers such that $\bar{\lambda}_j \neq \pm i \lambda_l$ for all $j, l = 1, \dots, n$. u is the n -soliton solution given by (38). Then for fixed t , there are positive constants r_0 , χ and C such that

$$|u(r \cos \theta, r \sin \theta, t)| \leq C e^{-\chi r} \quad (56)$$

for $r > r_0$ and all $e^{i\theta} \in S^1$. Hence $u(r \cos \theta, r \sin \theta, t) \rightarrow 0$ uniformly and exponentially as $r \rightarrow +\infty$.

Proof. The proof is divided into four steps.

Step 1: Obtain the asymptotic behavior of ξ_j 's and η_j 's.

Let $\lambda_j = \mu_j + i\nu_j$ where μ_j 's and ν_j 's are real, then $\mu_j \neq \pm \nu_j$ for all $j = 1, \dots, n$.

Let $Z^{(\varepsilon)} = \{e^{i\theta} \in S^1 \mid \tan \theta = \varepsilon\}$ for $\varepsilon = \pm 1$, $Z = Z^{(+1)} \cup Z^{(-1)}$. If $e^{i\theta} \in Z^{(\varepsilon)}$, then by (33), $\alpha_j^{(1)}(\theta) = \alpha_j^{(2)}(\theta) = \frac{1}{\sqrt{2}}|\mu_j - \varepsilon \nu_j| > 0$, $\varepsilon_j^{(2)}(\theta) = \varepsilon \varepsilon_j^{(1)}(\theta)$ and $\varepsilon_j^{(1)}(\theta)(\mu_j - \varepsilon \nu_j) \cos \theta > 0$ for all $j = 1, \dots, n$. If $e^{i\theta} \in S^1 \setminus Z$, then $\alpha_j^{(1)}(\theta) = |\mu_j \cos \theta - \nu_j \sin \theta|$, $\alpha_j^{(2)}(\theta) = |-\nu_j \cos \theta + \mu_j \sin \theta|$ with $\alpha_j^{(1)}(\theta) \neq \alpha_j^{(2)}(\theta)$.

For $\delta \in (0, \pi/4)$, define

$$\begin{aligned} \Omega_\delta^{(\varepsilon)} &= \left\{ e^{i\theta} \mid \text{there exists } e^{i\theta_0} \in Z^{(\varepsilon)} \text{ such that } |\theta - \theta_0| < \delta \right\} \quad (\varepsilon = \pm 1), \\ \Omega_\delta^{(0)} &= \left\{ e^{i\theta} \mid |\theta - \theta_0| > \delta/2 \text{ for all } e^{i\theta_0} \in Z \right\}. \end{aligned} \quad (57)$$

Then $\Omega_\delta^{(+1)} \cup \Omega_\delta^{(-1)} \cup \Omega_\delta^{(0)} = S^1$, and there exists $\delta \in (0, \pi/4)$ and $\omega > 0$ such that

$$\begin{aligned} \alpha_j^{(1)}(\theta) &> \omega, \quad \alpha_j^{(2)}(\theta) > \omega, \quad \varepsilon_j^{(2)}(\theta) = \varepsilon \varepsilon_j^{(1)}(\theta), \quad \varepsilon_j^{(1)}(\theta)(\mu_j - \varepsilon \nu_j) \cos \theta > 0 \quad \text{if } e^{i\theta} \in \Omega_\delta^{(\varepsilon)}, \\ |\alpha_j^{(1)}(\theta) - \alpha_j^{(2)}(\theta)| &> \omega \quad \text{if } e^{i\theta} \in \Omega_\delta^{(0)}. \end{aligned} \quad (58)$$

For $e^{i\theta_0} \in Z^{(\varepsilon)}$ and $e^{i\theta} \in \Omega_\delta^{(\varepsilon)}$ with $|\theta - \theta_0| < \delta$, $\varepsilon_j^{(1)}(\theta)$ is a constant,

$$\begin{aligned} &\alpha_j^{(1)}(\theta) - \alpha_j^{(2)}(\theta) \\ &= \varepsilon_j^{(1)}(\theta)(\mu_j \cos \theta - \nu_j \sin \theta) - \varepsilon_j^{(2)}(\theta)(-\nu_j \cos \theta + \mu_j \sin \theta) \\ &= \varepsilon \varepsilon_j^{(1)}(\theta)(\mu_j + \varepsilon \nu_j) \cos \theta (\varepsilon - \tan \theta). \end{aligned} \quad (59)$$

Hence, if $\alpha_j^{(1)}(\theta) > \alpha_j^{(2)}(\theta)$ for $e^{i\theta} \in \Omega_\delta^{(\varepsilon)} \setminus Z^{(\varepsilon)}$ with $0 < \theta - \theta_0 < \delta$, then $\alpha_j^{(1)}(\theta) < \alpha_j^{(2)}(\theta)$ for $e^{i\theta} \in \Omega_\delta^{(\varepsilon)} \setminus Z^{(\varepsilon)}$ with $-\delta < \theta - \theta_0 < 0$, and *vice versa*.

Recall that

$$\begin{aligned}\xi_j &= \kappa_j^{(1)}(e^{\varepsilon_j^{(1)}(\theta)\lambda_j x + i\varepsilon_j^{(1)}(\theta)\lambda_j y + 16i\varepsilon_j^{(1)}(\theta)\lambda_j^3 t} + e^{-\varepsilon_j^{(1)}(\theta)\lambda_j x - i\varepsilon_j^{(1)}(\theta)\lambda_j y - 16i\varepsilon_j^{(1)}(\theta)\lambda_j^3 t}) \\ &= \kappa_j^{(1)} e^{\alpha_j^{(1)}(\theta)r + \varepsilon_j^{(1)}(\theta)\beta_j^{(1)} + i\varepsilon_j^{(1)}(\theta)\sigma_j^{(1)}(\theta, r)} \left(1 + e^{-2\alpha_j^{(1)}(\theta)r - 2\varepsilon_j^{(1)}(\theta)\beta_j^{(1)} - 2i\varepsilon_j^{(1)}(\theta)\sigma_j^{(1)}(\theta, r)}\right), \\ \eta_j &= \kappa_j^{(2)}(e^{i\varepsilon_j^{(2)}(\theta)\lambda_j x + \varepsilon_j^{(2)}(\theta)\lambda_j y - 16\varepsilon_j^{(2)}(\theta)\lambda_j^3 t} + e^{-i\varepsilon_j^{(2)}(\theta)\lambda_j x - \varepsilon_j^{(2)}(\theta)\lambda_j y + 16\varepsilon_j^{(2)}(\theta)\lambda_j^3 t}) \\ &= \kappa_j^{(2)} e^{\alpha_j^{(2)}(\theta)r + \varepsilon_j^{(2)}(\theta)\beta_j^{(2)} + i\varepsilon_j^{(2)}(\theta)\sigma_j^{(2)}(\theta, r)} \left(1 + e^{-2\alpha_j^{(2)}(\theta)r - 2\varepsilon_j^{(2)}(\theta)\beta_j^{(2)} - 2i\varepsilon_j^{(2)}(\theta)\sigma_j^{(2)}(\theta, r)}\right).\end{aligned}\quad (60)$$

Let

$$Y_j(\theta, r) = (\kappa_j^{(1)})^{-1} \kappa_j^{(2)} e^{(\alpha_j^{(2)}(\theta) - \alpha_j^{(1)}(\theta))r + \varepsilon_j^{(2)}(\theta)\beta_j^{(2)} - \varepsilon_j^{(1)}(\theta)\beta_j^{(1)} + i\varepsilon_j^{(2)}(\theta)\sigma_j^{(2)}(\theta, r) - i\varepsilon_j^{(1)}(\theta)\sigma_j^{(1)}(\theta, r)}. \quad (61)$$

When $r \rightarrow +\infty$, the following limits hold uniformly.

For $e^{i\theta} \in \Omega_\delta^{(\varepsilon)}$ with $\alpha_j^{(1)}(\theta) \geq \alpha_j^{(2)}(\theta)$,

$$\xi_j^{-1} \partial^k \xi_j \rightarrow (\varepsilon_j^{(1)}(\theta)\lambda_j)^k, \quad \xi_j^{-1} \partial^k \eta_j - (i\varepsilon_j^{(2)}(\theta)\lambda_j)^k Y_j(\theta, r) \rightarrow 0. \quad (62)$$

For $e^{i\theta} \in \Omega_\delta^{(\varepsilon)}$ with $\alpha_j^{(1)}(\theta) \leq \alpha_j^{(2)}(\theta)$,

$$\eta_j^{-1} \partial^k \xi_j - (\varepsilon_j^{(1)}(\theta)\lambda_j)^k Y_j(\theta, r)^{-1} \rightarrow 0, \quad \eta_j^{-1} \partial^k \eta_j \rightarrow (i\varepsilon_j^{(2)}(\theta)\lambda_j)^k. \quad (63)$$

For $e^{i\theta} \in \Omega_\delta^{(0)}$ with $\alpha_j^{(1)}(\theta) > \alpha_j^{(2)}(\theta)$,

$$\xi_j^{-1} \partial^k \xi_j \rightarrow (\varepsilon_j^{(1)}(\theta)\lambda_j)^k, \quad \xi_j^{-1} \partial^k \eta_j \rightarrow 0. \quad (64)$$

For $e^{i\theta} \in \Omega_\delta^{(0)}$ with $\alpha_j^{(1)}(\theta) < \alpha_j^{(2)}(\theta)$,

$$\eta_j^{-1} \partial^k \xi_j \rightarrow 0, \quad \eta_j^{-1} \partial^k \eta_j \rightarrow (i\varepsilon_j^{(2)}(\theta)\lambda_j)^k. \quad (65)$$

Step 2: There exists $r_0 > 0$ and $c_0 > 0$ such that $\det \tilde{T} > c_0$ when $r > r_0$.

When $e^{i\theta} \in Z^{(\varepsilon)}$ ($\varepsilon = \pm 1$), $\alpha_j^{(1)}(\theta) = \alpha_j^{(2)}(\theta)$ for $j = 1, \dots, n$. (60) implies

$$\frac{|\eta_j(\theta, r)|}{|\xi_j(\theta, r)|} \rightarrow \gamma_j(\theta) \equiv \frac{|\kappa_j^{(2)}| e^{\varepsilon_j^{(2)}(\theta)\beta_j^{(2)}}}{|\kappa_j^{(1)}| e^{\varepsilon_j^{(1)}(\theta)\beta_j^{(1)}}} \quad (66)$$

as $r \rightarrow +\infty$. By (62),

$$\det \tilde{T}(\theta, r) = \left(\prod_{|\gamma_j(\theta)| > 1} |\gamma_j(\theta)| \right)^{-2} \left| \begin{array}{cc} A(\theta, r) & B(\theta, r) \\ -\bar{B}(\theta, r) & \bar{A}(\theta, r) \end{array} \right| + o(1), \quad (67)$$

where $A(\theta, r)$ and $B(\theta, r)$ are $n \times n$ matrices, whose entries are

$$A_{jk}(\theta, r) = (\varepsilon_j^{(1)}(\theta)\lambda_j)^k, \quad B_{jk}(\theta, r) = (i\varepsilon_j^{(2)}(\theta)\lambda_j)^k Y_j(\theta, r) = (i\varepsilon_j^{(1)}(\theta)\lambda_j)^k Y_j(\theta, r) \quad (68)$$

and $o(1)$ refers to the terms which tend to zero as $r \rightarrow +\infty$. Let

$$\begin{aligned}\Lambda &= (\varepsilon_1^{(1)}(\theta)\lambda_1, \dots, \varepsilon_n^{(1)}(\theta)\lambda_n), \\ \Gamma &= \text{diag}(Y_1(\theta, r), \dots, Y_n(\theta, r)).\end{aligned}\tag{69}$$

By Lemma 3 of A, there exist $r_1 > 0$ and $c_1 > 0$ such that $\det \tilde{T} > c_1$ for $e^{i\theta} \in Z^{(\varepsilon)}$ and $r > r_1$.

When $e^{i\theta} \in \Omega_\delta^{(\varepsilon)} \setminus Z$ ($\varepsilon = \pm 1$) with $0 < \theta - \theta_0 < \delta$ ($\theta_0 \in Z^{(\varepsilon)}$), $\varepsilon_j^{(2)}(\theta) = \varepsilon \varepsilon_j^{(1)}(\theta)$. Suppose $\alpha_j^{(1)}(\theta) > \alpha_j^{(2)}(\theta)$ for $j = 1, \dots, m$ and $\alpha_j^{(1)}(\theta) < \alpha_j^{(2)}(\theta)$ for $j = m+1, \dots, n$. By (62) and (63),

$$\det \tilde{T}(\theta, r) = \begin{vmatrix} A(\theta) & B(\theta, r) \\ C(\theta, r) & D(\theta) \\ -\bar{B}(\theta, r) & \bar{A}(\theta) \\ -\bar{D}(\theta) & \bar{C}(\theta, r) \end{vmatrix} + o(1) = \begin{vmatrix} A(\theta) & B(\theta, r) \\ \bar{D}(\theta) & -\bar{C}(\theta, r) \\ -\bar{B}(\theta, r) & \bar{A}(\theta) \\ C(\theta, r) & D(\theta) \end{vmatrix} + o(1)\tag{70}$$

where $A(\theta)$ and $B(\theta, r)$ are $m \times n$ matrices, $C(\theta, r)$ and $D(\theta)$ are $(n-m) \times n$ matrices, whose entries are given by

$$\begin{aligned}A_{jk}(\theta) &= (\varepsilon_j^{(1)}(\theta)\lambda_j)^k, \quad B_{jk}(\theta, r) = (i\varepsilon_j^{(1)}(\theta)\lambda_j)^k Y_j(\theta, r) \quad (j = 1, \dots, m), \\ \bar{D}_{jk}(\theta) &= (-i\varepsilon_j^{(2)}(\theta)\bar{\lambda}_j)^k, \quad -\bar{C}_{jk}(\theta, r) = (\varepsilon \varepsilon_j^{(2)}(\theta)\bar{\lambda}_j)^k (-\bar{Y}_j(\theta, r)^{-1}) \quad (j = m+1, \dots, n)\end{aligned}\tag{71}$$

and $o(1)$ refers to the terms which tend to zero uniformly as $r \rightarrow +\infty$.

Let

$$\begin{aligned}\Lambda &= (\varepsilon_1^{(1)}(\theta)\lambda_1, \dots, \varepsilon_m^{(1)}(\theta)\lambda_m, -i\varepsilon_{m+1}^{(2)}(\theta)\bar{\lambda}_{m+1}, \dots, -i\varepsilon_n^{(2)}(\theta)\bar{\lambda}_n), \\ \Gamma &= \text{diag}(Y_1(\theta, r), \dots, Y_m(\theta, r), -\bar{Y}_{m+1}(\theta, r)^{-1}, \dots, -\bar{Y}_n(\theta, r)^{-1}).\end{aligned}\tag{72}$$

By Lemma 3 of A, there exist $r_2 > 0$ and $c_2 > 0$ such that $\det \tilde{T} > c_2$ for $e^{i\theta} \in \Omega_\delta^{(+1)} \cup \Omega_\delta^{(-1)} \setminus Z$ with $0 < \theta - \theta_0 < \delta$ and $r > r_2$.

Similarly, when $e^{i\theta} \in \Omega_\delta^{(+1)} \cup \Omega_\delta^{(-1)} \setminus Z$ with $-\delta < \theta - \theta_0 < 0$ ($\theta_0 \in Z$), there exist $r_3 > 0$ and $c_3 > 0$ such that $\det \tilde{T} > c_3$ for $r > r_3$.

When $e^{i\theta} \in \Omega_\delta^{(0)}$, suppose $\alpha_j^{(1)}(\theta) > \alpha_j^{(2)}(\theta)$ for $j = 1, \dots, m$ and $\alpha_j^{(1)}(\theta) < \alpha_j^{(2)}(\theta)$ for $j = m+1, \dots, n$, then, by (64) and (65),

$$\lim_{r \rightarrow +\infty} \det \tilde{T}(\theta, r) = \begin{vmatrix} A(\theta) & 0 \\ 0 & D(\theta) \\ 0 & \bar{A}(\theta) \\ -\bar{D}(\theta) & 0 \end{vmatrix} = \begin{vmatrix} A(\theta) & 0 \\ \bar{D}(\theta) & 0 \\ 0 & \bar{A}(\theta) \\ 0 & D(\theta) \end{vmatrix} = \left| \begin{vmatrix} A(\theta) \\ \bar{D}(\theta) \end{vmatrix} \right|^2\tag{73}$$

holds uniformly, where $A(\theta)$ is an $m \times n$ matrix, $D(\theta)$ is an $(n-m) \times n$ matrix, whose entries are given by

$$\begin{aligned}A_{jk}(\theta) &= (\varepsilon_j^{(1)}(\theta)\lambda_j)^k, \quad (j = 1, \dots, m), \\ \bar{D}_{jk}(\theta) &= (-i\varepsilon_j^{(2)}(\theta)\bar{\lambda}_j)^k \quad (j = m+1, \dots, n).\end{aligned}\tag{74}$$

Using the condition $\bar{\lambda}_j \neq \pm i\lambda_l$ and the property of Vandermonde determinant, we know that there exist $r_4 > 0$ and $c_4 > 0$ such that $\det \tilde{T} > c_4$ for all $e^{i\theta} \in \Omega_\delta^{(0)}$ and $r > r_4$.

Let $r_0 = \max(r_1, r_2, r_3, r_4)$, $c_0 = \min(c_1, c_2, c_3, c_4)$, then for any $e^{i\theta} \in S^1$, $\det \tilde{T} > c_0$ when $r > r_0$.

Step 3: Denote $\tilde{\Pi} = \begin{pmatrix} \overset{\circ}{T} & \\ & \overset{\circ}{T} \end{pmatrix}^{-1} \Pi$, then $\lim_{r \rightarrow +\infty} \operatorname{Re} \det \tilde{\Pi} = 0$ for any fixed $e^{i\theta} \in S^1$.

When $e^{i\theta} \in Z^{(\varepsilon)}$, considering (62), (63) and $\varepsilon_j^{(2)}(\theta) = \varepsilon \varepsilon_j^{(1)}(\theta)$, let $\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2)$ with

$$\begin{aligned} \Lambda_1 &= \operatorname{diag}(\varepsilon_1^{(1)}(\theta)\lambda_1, \dots, \varepsilon_n^{(1)}(\theta)\lambda_n), \\ \Lambda_2 &= \operatorname{diag}(-i\varepsilon\varepsilon_1^{(1)}(\theta)\bar{\lambda}_1, \dots, -i\varepsilon\varepsilon_n^{(1)}(\theta)\bar{\lambda}_n), \end{aligned} \quad (75)$$

$$\zeta = (\underbrace{1, \dots, 1}_n, -\bar{\xi}_1^{-1}\bar{\eta}_1, \dots, -\bar{\xi}_n^{-1}\bar{\eta}_n)^T, \quad (76)$$

then $\bar{\Lambda}_2 = i\varepsilon\Lambda_1$, and $\tilde{\Pi} - \left(\prod_{|\gamma_j(\theta)| > 1} |\gamma_j(\theta)| \right)^{-4} \Pi^\Lambda \rightarrow 0$ as $r \rightarrow +\infty$ where $\gamma_j(\theta)$'s are defined by (66) and Π^Λ is defined by (122). According to Lemma 4 of A, $\operatorname{Re} \det \Pi^\Lambda \equiv 0$, which leads to $\lim_{r \rightarrow +\infty} \operatorname{Re} \det \tilde{\Pi} = 0$.

When $e^{i\theta} \in S^1 \setminus Z$, suppose $\alpha_j^{(1)}(\theta) > \alpha_j^{(2)}(\theta)$ for $j = 1, \dots, m$ and $\alpha_j^{(1)}(\theta) < \alpha_j^{(2)}(\theta)$ for $j = m+1, \dots, n$. By (64) and (65),

$$\begin{aligned} \xi_j^{-1}(\partial^k \xi_j - (\varepsilon_j^{(1)}(\theta)\lambda_j)^k \xi_j) &\rightarrow 0, \quad \xi_j^{-1}(\partial^k \eta_j - (i\varepsilon_j^{(1)}(\theta)\lambda_j)^k \eta_j) \rightarrow 0 \quad (j = 1, \dots, m), \\ \eta_j^{-1}(\partial^k \xi_j - (\varepsilon_j^{(2)}(\theta)\lambda_j)^k \xi_j) &\rightarrow 0, \quad \eta_j^{-1}(\partial^k \eta_j - (i\varepsilon_j^{(2)}(\theta)\lambda_j)^k \eta_j) \rightarrow 0 \quad (j = m+1, \dots, n) \end{aligned} \quad (77)$$

as $r \rightarrow +\infty$ since $\xi_j^{-1}\eta_j \rightarrow 0$ for $j = 1, \dots, m$ and $\eta_j^{-1}\xi_j \rightarrow 0$ for $j = m+1, \dots, n$.

Let $\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2)$ with

$$\begin{aligned} \Lambda_1 &= \operatorname{diag}(\varepsilon_1^{(1)}(\theta)\lambda_1, \dots, \varepsilon_m^{(1)}(\theta)\lambda_m, \varepsilon_{m+1}^{(2)}(\theta)\lambda_{m+1}, \dots, \varepsilon_n^{(2)}(\theta)\lambda_n), \\ \Lambda_2 &= \operatorname{diag}(-i\varepsilon_1^{(1)}(\theta)\bar{\lambda}_1, \dots, -i\varepsilon_m^{(1)}(\theta)\bar{\lambda}_m, -i\varepsilon_{m+1}^{(2)}(\theta)\bar{\lambda}_{m+1}, \dots, -i\varepsilon_n^{(2)}(\theta)\bar{\lambda}_n), \end{aligned} \quad (78)$$

$$\zeta = (\underbrace{1, \dots, 1}_m, \underbrace{0, \dots, 0}_{n-m}, \underbrace{0, \dots, 0}_m, \underbrace{-1, \dots, -1}_{n-m})^T, \quad (79)$$

then $\bar{\Lambda}_2 = i\Lambda_1$, and $\tilde{\Pi} \rightarrow \Pi^\Lambda$ as $r \rightarrow +\infty$ where Π^Λ is defined by (122). According to Lemma 4 in A, we have $\lim_{r \rightarrow +\infty} \operatorname{Re} \det \tilde{\Pi} = \operatorname{Re} \det \Pi^\Lambda = 0$.

Till now, we have proved that $\det \tilde{T}$ has a uniform positive lower bound for all θ , and $\lim_{r \rightarrow +\infty} \frac{\operatorname{Re} \det \Pi}{(\det T)^2} = 0$ for any fixed θ .

Step 4: $\lim_{r \rightarrow +\infty} \frac{\operatorname{Re} \det \Pi}{(\det T)^2} = 0$ uniformly for all $e^{i\theta} \in S^1$ as $r \rightarrow +\infty$.

Note that $\frac{\operatorname{Re} \det \Pi}{(\det T)^2}$ is of form $\frac{f(\theta, r)}{g(\theta, r)}$ where

$$f(\theta, r) = \sum_{j=1}^{m_1} e^{\tilde{\alpha}_j(\theta)r + \tilde{\gamma}_j(\theta)}, \quad g(\theta, r) = \sum_{j=1}^{m_2} e^{\tilde{\beta}_j(\theta)r + \tilde{\delta}_j(\theta)} \quad (80)$$

are real-valued functions of (θ, r) , $\tilde{\alpha}_j(\theta)$, $\tilde{\beta}_j(\theta)$, $\tilde{\gamma}_j(\theta)$, $\tilde{\delta}_j(\theta)$ are (complex valued) continuous functions of θ . Let $\tilde{a}(\theta) = \max_{1 \leq j \leq m_1} \operatorname{Re} \tilde{\alpha}_j(\theta)$, $\tilde{b}(\theta) = \max_{1 \leq j \leq m_2} \operatorname{Re} \tilde{\beta}_j(\theta)$. Since $\lim_{r \rightarrow +\infty} \frac{f(\theta, r)}{g(\theta, r)} = 0$ for any fixed θ , the real continuous function $\tilde{a}(\theta) - \tilde{b}(\theta) < 0$ achieves its maximum $-\chi < 0$ on the compact set S^1 . We have known that $\det \tilde{T}$ has a uniform positive lower bound as $r \geq r_0$ (r_0 is independent of θ), so has

$$\sum_{j=1}^{m_2} e^{(\tilde{\beta}_j(\theta) - \tilde{b}(\theta))r + \tilde{\delta}_j(\theta)}. \quad (81)$$

Hence

$$\left| \frac{f(\theta, r)}{g(\theta, r)} \right| \leq e^{-\chi r} \frac{\left| \sum_{j=1}^{m_1} e^{(\tilde{\alpha}_j(\theta) - \tilde{a}(\theta))r + \tilde{\gamma}_j(\theta)} \right|}{\sum_{j=1}^{m_2} e^{(\tilde{\beta}_j(\theta) - \tilde{b}(\theta))r + \tilde{\delta}_j(\theta)}} \leq C e^{-\chi r} \quad (82)$$

as $r \geq r_0$ where C is a constant independent of θ . The theorem is proved.

7 Asymptotic behavior of the solutions as $t \rightarrow \infty$

In this section, the asymptotic behavior of the n -soliton solutions as $t \rightarrow \infty$ will be discussed. In order to do so, we consider the problem in a moving frame. Let $x = x_0 + \theta_1 t$, $y = y_0 + \theta_2 t$ where (θ_1, θ_2) is the velocity of the moving frame, and (x_0, y_0) is the coordinate in the moving frame with this velocity. Then

$$\begin{aligned} \rho_j^{(1)}(x_0 + \theta_1 t, y_0 + \theta_2 t, t) &= \varepsilon_j^{(1)} \alpha_j^{(1)} t + \beta_j^{(1)}, \\ \rho_j^{(2)}(x_0 + \theta_1 t, y_0 + \theta_2 t, t) &= \varepsilon_j^{(2)} \alpha_j^{(2)} t + \beta_j^{(2)} \end{aligned} \quad (83)$$

where $\varepsilon_j^{(k)} = \pm 1$ ($j = 1, \dots, n$; $k = 1, 2$) so that $\alpha_j^{(k)} \geq 0$. Write $\lambda_j = \mu_j + i\nu_j$ ($j = 1, \dots, n$) where μ_j 's and ν_j 's are real, then according to (33),

$$\begin{aligned} \varepsilon_j^{(1)} \alpha_j^{(1)} &= \mu_j \theta_1 - \nu_j \theta_2 + 16(\nu_j^3 - 3\mu_j^2 \nu_j), \\ \varepsilon_j^{(2)} \alpha_j^{(2)} &= -\nu_j \theta_1 + \mu_j \theta_2 - 16(\mu_j^3 - 3\mu_j \nu_j^2), \end{aligned} \quad (84)$$

and

$$\begin{aligned} \varepsilon_j^{(1)} \alpha_j^{(1)} - \varepsilon_j^{(2)} \alpha_j^{(2)} &= (\mu_j + \nu_j)(\theta_1 - \theta_2 + 16(\mu_j^2 - 4\mu_j \nu_j + \nu_j^2)), \\ \varepsilon_j^{(1)} \alpha_j^{(1)} + \varepsilon_j^{(2)} \alpha_j^{(2)} &= (\mu_j - \nu_j)(\theta_1 + \theta_2 - 16(\mu_j^2 + 4\mu_j \nu_j + \nu_j^2)). \end{aligned} \quad (85)$$

Theorem 3 Suppose $\lambda_j = \mu_j + i\nu_j \neq 0$ ($j = 1, \dots, n$) are distinct complex numbers where μ_j 's and ν_j 's are real numbers, such that

$$\begin{aligned} \mu_j &\neq \pm \nu_j \quad \text{for all } j, \\ \mu_j^2 + 4\mu_j \nu_j + \nu_j^2 &\neq \mu_l^2 + 4\mu_l \nu_l + \nu_l^2 \quad \text{for all } j \neq l, \\ \mu_j^2 - 4\mu_j \nu_j + \nu_j^2 &\neq \mu_l^2 - 4\mu_l \nu_l + \nu_l^2 \quad \text{for all } j \neq l. \end{aligned} \quad (86)$$

u is the n -soliton solution given by (38). Then for bounded (x_0, y_0) , $\lim_{t \rightarrow \infty} u(x_0 + \theta_1 t, y_0 + \theta_2 t, t) = 0$ except when

$$\begin{aligned}\theta_1 &= 8(\mu_l^2 + 4\mu_l\nu_l + \nu_l^2 - \mu_j^2 + 4\mu_j\nu_j - \nu_j^2), \\ \theta_2 &= 8(\mu_l^2 + 4\mu_l\nu_l + \nu_l^2 + \mu_j^2 - 4\mu_j\nu_j + \nu_j^2), \\ (j, l &= 1, 2, \dots, n).\end{aligned}\tag{87}$$

Therefore, as $t \rightarrow \infty$, u has at most $n \times n$ lumps of peaks which move in the above velocities (θ_1, θ_2) respectively.

Proof.

We will always suppose that (θ_1, θ_2) does not satisfy (87). Then, by (85), $\alpha_j^{(1)} \neq 0$ whenever $\alpha_j^{(1)} = \alpha_j^{(2)}$. Moreover, we only consider the limit $t \rightarrow +\infty$. The conclusion is the same for $t \rightarrow -\infty$.

The proof is divided into three steps.

Step 1: Obtain the asymptotic behavior of ξ_j 's and η_j 's.

Suppose $\alpha_j^{(1)} > \alpha_j^{(2)}$ for $j = 1, \dots, m$; $\alpha_j^{(1)} < \alpha_j^{(2)}$ for $j = m+1, \dots, p$; $\alpha_j^{(1)} = \alpha_j^{(2)} \neq 0$ for $j = p+1, \dots, n$. Then

$$\begin{aligned}\xi_j^{-1}(\partial^k \xi_j - (\varepsilon_j^{(1)} \lambda_j)^k \xi_j) &\rightarrow 0, \quad \xi_j^{-1}(\partial^k \eta_j - (i\varepsilon_j^{(2)} s_j \lambda_j)^k \eta_j) \rightarrow 0 \quad (j = 1, \dots, m), \\ \eta_j^{-1}(\partial^k \xi_j - (\varepsilon_j^{(1)} s_j \lambda_j)^k \xi_j) &\rightarrow 0, \quad \eta_j^{-1}(\partial^k \eta_j - (i\varepsilon_j^{(2)} \lambda_j)^k \eta_j) \rightarrow 0 \quad (j = m+1, \dots, p), \\ \xi_j^{-1}(\partial^k \xi_j - (\varepsilon_j^{(1)} \lambda_j)^k \xi_j) &\rightarrow 0, \quad \xi_j^{-1}(\partial^k \eta_j - (i\varepsilon_j^{(2)} \lambda_j)^k \eta_j) \rightarrow 0 \quad (j = p+1, \dots, n)\end{aligned}\tag{88}$$

as $r \rightarrow +\infty$ where s_1, \dots, s_p are any constants, since $\xi_j^{-1} \eta_j \rightarrow 0$ for $j = 1, \dots, m$ and $\eta_j^{-1} \xi_j \rightarrow 0$ for $j = m+1, \dots, p$.

Now we prove that p can only take $n-1$ or n . If $p \leq n-2$, then $\varepsilon_j^{(1)} \varepsilon_j^{(2)} = \pm \varepsilon_l^{(1)} \varepsilon_l^{(2)}$ must hold for any $j \neq l$ with $p+1 \leq j, l \leq n$ since both sides equal ± 1 . If $\varepsilon_j^{(1)} \varepsilon_j^{(2)} = \varepsilon_l^{(1)} \varepsilon_l^{(2)}$, then $\varepsilon_j^{(2)} \alpha_j^{(2)} = \varepsilon \varepsilon_j^{(1)} \alpha_j^{(1)}$, $\varepsilon_l^{(2)} \alpha_l^{(2)} = \varepsilon \varepsilon_l^{(1)} \alpha_l^{(1)}$ hold simultaneously where $\varepsilon = \varepsilon_j^{(1)} \varepsilon_j^{(2)}$. This contradicts condition (86). If $\varepsilon_j^{(1)} \varepsilon_j^{(2)} = -\varepsilon_l^{(1)} \varepsilon_l^{(2)}$, then $\varepsilon_j^{(2)} \alpha_j^{(2)} - \varepsilon \varepsilon_j^{(1)} \alpha_j^{(1)} = 0$, $\varepsilon_l^{(2)} \alpha_l^{(2)} + \varepsilon \varepsilon_l^{(1)} \alpha_l^{(1)} = 0$ hold simultaneously where $\varepsilon = \varepsilon_j^{(1)} \varepsilon_j^{(2)}$. This contradicts the assumption that (θ_1, θ_2) does not satisfy (87). Hence only $p = n$ or $p = n-1$ is possible.

Step 2: There exists $t_0 > 0$ and $c > 0$ such that $\det \tilde{T} > c$ for $t \geq t_0$.

When $p = n-1$,

$$\frac{|\xi_n(\theta, r)|}{\max(|\xi_n(\theta, r)|, |\eta_n(\theta, r)|)} \rightarrow \gamma_0 \equiv \frac{|\kappa_n^{(1)}| e^{\varepsilon_n^{(1)} \beta_n^{(1)}}}{\max(|\kappa_n^{(1)}| e^{\varepsilon_n^{(1)} \beta_n^{(1)}}, |\kappa_n^{(2)}| e^{\varepsilon_n^{(2)} \beta_n^{(2)}})}.\tag{89}$$

Denote

$$V(\lambda_j, \dots, \lambda_l) = \begin{pmatrix} \lambda_j^{n-1} & \dots & 1 \\ \vdots & & \vdots \\ \lambda_l^{n-1} & \dots & 1 \end{pmatrix}_{(l-j+1) \times n}\tag{90}$$

for $j \leq l$, then, for $p = n - 1$, (88) leads to

$$\begin{aligned}
\det \tilde{T} &= \gamma_0^2 \begin{vmatrix} V(\varepsilon_1^{(1)} \lambda_1, \dots, \varepsilon_m^{(1)} \lambda_m) & 0 \\ 0 & V(\mathrm{i} \varepsilon_{m+1}^{(2)} \lambda_{m+1}, \dots, \mathrm{i} \varepsilon_{n-1}^{(2)} \lambda_{n-1}) \\ V(\varepsilon_n^{(1)} \lambda_n) & \xi_n^{-1} \eta_n V(\mathrm{i} \varepsilon_n^{(2)} \lambda_n) \\ 0 & V(\varepsilon_1^{(1)} \bar{\lambda}_1, \dots, \varepsilon_m^{(1)} \bar{\lambda}_m) \\ -V(-\mathrm{i} \varepsilon_{m+1}^{(2)} \bar{\lambda}_{m+1}, \dots, -\mathrm{i} \varepsilon_{n-1}^{(2)} \bar{\lambda}_{n-1}) & 0 \\ -\bar{\xi}_n^{-1} \bar{\eta}_n V(-\mathrm{i} \varepsilon_n^{(2)} \bar{\lambda}_n) & V(\varepsilon_n^{(1)} \bar{\lambda}_n) \end{vmatrix} + o(1) \\
&= \gamma_0^2 \begin{vmatrix} V(\varepsilon_1^{(1)} \lambda_1, \dots, \varepsilon_m^{(1)} \lambda_m) & 0 \\ V(-\mathrm{i} \varepsilon_{m+1}^{(2)} \bar{\lambda}_{m+1}, \dots, -\mathrm{i} \varepsilon_{n-1}^{(2)} \bar{\lambda}_{n-1}) & 0 \\ V(\varepsilon_n^{(1)} \lambda_n) & \xi_n^{-1} \eta_n V(\mathrm{i} \varepsilon_n^{(2)} \lambda_n) \\ 0 & V(\varepsilon_1^{(1)} \bar{\lambda}_1, \dots, \varepsilon_m^{(1)} \bar{\lambda}_m) \\ 0 & V(\mathrm{i} \varepsilon_{m+1}^{(2)} \lambda_{m+1}, \dots, \mathrm{i} \varepsilon_{n-1}^{(2)} \lambda_{n-1}) \\ -\bar{\xi}_n^{-1} \bar{\eta}_n V(-\mathrm{i} \varepsilon_n^{(2)} \bar{\lambda}_n) & V(\varepsilon_n^{(1)} \bar{\lambda}_n) \end{vmatrix} + o(1)
\end{aligned} \tag{91}$$

as $t \rightarrow +\infty$. Let

$$\begin{aligned}
\Lambda &= \text{diag}(\varepsilon_1^{(1)} \lambda_1, \dots, \varepsilon_m^{(1)} \lambda_m, -\mathrm{i} \varepsilon_{m+1}^{(2)} \bar{\lambda}_{m+1}, \dots, -\mathrm{i} \varepsilon_{n-1}^{(2)} \bar{\lambda}_{n-1}, \varepsilon_n^{(1)} \lambda_n), \\
\Gamma &= \text{diag}(0, \dots, 0, \xi_n^{-1} \eta_n), \quad \varepsilon = \varepsilon_n^{(1)} \varepsilon_n^{(2)},
\end{aligned} \tag{92}$$

then we get $\liminf_{t \rightarrow +\infty} \det \tilde{T} > 0$ by Lemma 3 of A.

Similarly, when $p = n$,

$$\begin{aligned}
\lim_{t \rightarrow +\infty} \det \tilde{T} &= \begin{vmatrix} V(\varepsilon_1^{(1)} \lambda_1, \dots, \varepsilon_m^{(1)} \lambda_m) & 0 \\ V(-\mathrm{i} \varepsilon_{m+1}^{(2)} \bar{\lambda}_{m+1}, \dots, -\mathrm{i} \varepsilon_n^{(2)} \bar{\lambda}_n) & 0 \\ 0 & V(\varepsilon_1^{(1)} \bar{\lambda}_1, \dots, \varepsilon_m^{(1)} \bar{\lambda}_m) \\ 0 & V(\mathrm{i} \varepsilon_{m+1}^{(2)} \lambda_{m+1}, \dots, \mathrm{i} \varepsilon_n^{(2)} \lambda_n) \end{vmatrix} \\
&= \left| \det V(\varepsilon_1^{(1)} \lambda_1, \dots, \varepsilon_m^{(1)} \lambda_m, -\mathrm{i} \varepsilon_{m+1}^{(2)} \bar{\lambda}_{m+1}, \dots, -\mathrm{i} \varepsilon_n^{(2)} \bar{\lambda}_n) \right|^2.
\end{aligned} \tag{93}$$

Using the condition $\bar{\lambda}_j \neq \pm \mathrm{i} \lambda_l$, we get $\liminf_{t \rightarrow +\infty} \det \tilde{T} > 0$ since the Vandermonde determinant is non-zero.

Step 3: Denote $\tilde{\Pi} = \begin{pmatrix} \mathring{T} & \\ & \mathring{T} \end{pmatrix}^{-1} \Pi$. If the velocity (θ_1, θ_2) does not satisfy (87), then

$$\lim_{t \rightarrow +\infty} \text{Re} \det \tilde{\Pi} = 0.$$

From (88), let $\Lambda = \text{diag}(\Lambda_1, \Lambda_2)$ with

$$\begin{aligned}
\Lambda_1 &= \text{diag}(\varepsilon_1^{(1)} \lambda_1, \dots, \varepsilon_m^{(1)} \lambda_m, \varepsilon_{m+1}^{(1)} s_{m+1} \lambda_{m+1}, \dots, \varepsilon_p^{(1)} s_p \lambda_p, \varepsilon_{p+1}^{(1)} \lambda_{p+1}, \dots, \varepsilon_n^{(1)} \lambda_n), \\
\Lambda_2 &= \text{diag}(-\mathrm{i} \varepsilon_1^{(2)} \bar{s}_1 \bar{\lambda}_1, \dots, -\mathrm{i} \varepsilon_m^{(2)} \bar{s}_m \bar{\lambda}_m, -\mathrm{i} \varepsilon_{m+1}^{(2)} \bar{\lambda}_{m+1}, \dots, -\mathrm{i} \varepsilon_p^{(2)} \bar{\lambda}_p, \\
&\quad -\mathrm{i} \varepsilon_{p+1}^{(2)} \bar{\lambda}_{p+1}, \dots, -\mathrm{i} \varepsilon_n^{(2)} \bar{\lambda}_n),
\end{aligned} \tag{94}$$

Figure 4: 3×3 soliton solution u : $\lambda_1 = 2 + 0.5i$, $\lambda_2 = 2.5 + 0.4i$, $\lambda_3 = 3 + 0.3i$, $\kappa_j^{(1)} = 1$, $\kappa_j^{(2)} = 1.2$, $\rho_{j0}^{(1)} = 0$, $\rho_{j0}^{(2)} = 0$, $\sigma_{j0}^{(1)} = 0$, $\sigma_{j0}^{(2)} = 0$ ($j = 1, 2, 3$), $t = 1$.

$$\zeta = (\underbrace{1, \dots, 1}_m, \underbrace{0, \dots, 0}_{p-m}, \underbrace{1, \dots, 1}_{n-p}, \underbrace{0, \dots, 0}_m, \underbrace{-1, \dots, -1}_{p-m}, -\bar{\xi}_{p+1}^{-1} \bar{\eta}_{p+1}, \dots, -\bar{\xi}_n^{-1} \bar{\eta}_n)^T, \quad (95)$$

then $\tilde{\Pi} - \gamma_0^4 \Pi^\Lambda \rightarrow 0$ for $p = n - 1$ and $\tilde{\Pi} - \Pi^\Lambda \rightarrow 0$ for $p = n$ as $t \rightarrow +\infty$ where γ_0 is defined by (89) and Π^Λ is defined by (122). $\bar{\Lambda}_2 = i\varepsilon \Lambda_1$ holds for $\varepsilon = \pm 1$ if and only if $s_j = \varepsilon \varepsilon_j^{(1)} \varepsilon_j^{(2)}$ for $j = 1, \dots, p$, and $\varepsilon_j^{(2)} = \varepsilon \varepsilon_j^{(1)}$ for $j = p+1, \dots, n$. Since s_j ($j = 1, \dots, p$) can be arbitrary, ε can be taken as ± 1 arbitrarily, and $p = n$ or $n - 1$, we have $\bar{\Lambda}_2 = i\varepsilon \Lambda_1$ by taking $\varepsilon = \varepsilon_n^{(1)} \varepsilon_n^{(2)}$ for $p = n - 1$, and either $\varepsilon = 1$ or $\varepsilon = -1$ for $p = n$.

According to Lemma 4 of A, $\text{Re det } \Pi^\Lambda \equiv 0$. Hence $\lim_{r \rightarrow +\infty} \text{Re det } \tilde{\Pi} = 0$ if (θ_1, θ_2) does not satisfy (87). The theorem is proved.

Remark 2 If (θ_1, θ_2) satisfies (87), then $\alpha_j^{(1)} = \alpha_j^{(2)}$, $\alpha_l^{(1)} = \alpha_l^{(2)}$, $\varepsilon_j^{(2)} = \varepsilon_j^{(1)}$, $\varepsilon_l^{(2)} = -\varepsilon_l^{(1)}$ for $j \neq l$ with $p+1 \leq j, l \leq n$. Hence there is no common $\varepsilon = \pm 1$ such that $\varepsilon_i^{(2)} = \varepsilon \varepsilon_i^{(1)}$ holds for all $i = p+1, \dots, n$, which contradicts the condition $\bar{\Lambda}_2 = i\varepsilon \Lambda_1$ in Lemma 4 of A. In fact, the solution does not tend to zero in this case, which can be seen in the following example.

As an example, a 3×3 soliton is shown in Figure 4, in which there are 9 lumps of peaks. The local behavior of each lump of peaks is still complicated and one of which is shown in Figure 5.

A Some linear algebraic lemmas

Lemma 1 Suppose X and Y are $2n \times r$ and $2n \times (2n - r)$ matrices respectively, then

$$\left| \begin{array}{cc} X & K_n \bar{Y} \end{array} \right| = \left| \begin{array}{cc} Y & K_n \bar{X} \end{array} \right| \quad (96)$$

Figure 5: Local behavior of one lump of peaks in the 3×3 soliton solution u .

where $K_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

Proof.

$$K_n \overline{\begin{pmatrix} X & K_n \bar{Y} \end{pmatrix}} K_n^{-1} = \begin{pmatrix} K_n \bar{X} & -Y \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} Y & K_n \bar{X} \end{pmatrix}. \quad (97)$$

The lemma is obtained by taking the determinants on both sides.

Lemma 2 Suppose $T = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ where A and B are $n \times n$ matrices, then $\det T \geq 0$.

Proof. By Lemma 1, $\det T$ is real. First suppose both A and B are invertible, then

$$\det T = \det A \det(\bar{A} + \bar{B} A^{-1} B) = |\det A|^2 \det(I + \overline{A^{-1} B} A^{-1} B). \quad (98)$$

Let $N = A^{-1} B$. Suppose λ is an eigenvalue of $\bar{N} N$, $v \in \mathbf{R}^{2n}$ is a vector in the corresponding root space \mathcal{R}_λ , i.e. $(\bar{N} N - \lambda I)^m v = 0$ for certain positive integer m . Then

$$(\bar{N} N - \bar{\lambda} I)^m (\bar{N} \bar{v}) = \bar{N} \overline{(\bar{N} N - \lambda I)^m v} = 0. \quad (99)$$

Hence $\bar{N} \bar{v} \in \mathcal{R}_{\bar{\lambda}}$. If λ is a non-real eigenvalue of $\bar{N} N$ of multiplicity k , the multiplicity of $\bar{\lambda}$ is also k .

Now suppose $\lambda < 0$ is an eigenvalue of $\bar{N} N$, $\mathcal{R}_\lambda = V_1 \oplus \cdots \oplus V_m$ where V_1, \dots, V_m are irreducible invariant subspaces. Suppose $V_1 = \text{span}\{\zeta, (\bar{N} N - \lambda I)\zeta, \dots, (\bar{N} N - \lambda I)^{m-1}\zeta\}$ with $(\bar{N} N - \lambda I)^m \zeta = 0$. Then $\zeta \notin \text{Image}(\bar{N} N - \lambda I)$, and $(\bar{N} N - \lambda I)^m \bar{N} \bar{\zeta} = \bar{N} \overline{(\bar{N} N - \lambda I)^m \zeta} = 0$, $(\bar{N} N - \lambda I)^{m-1} \bar{N} \bar{\zeta} = \bar{N} \overline{(\bar{N} N - \lambda I)^{m-1} \zeta} \neq 0$ since $\det N \neq 0$. We will prove that

$$\zeta, (\bar{N} N - \lambda I)\zeta, \dots, (\bar{N} N - \lambda I)^{m-1}\zeta, \bar{N} \bar{\zeta}, (\bar{N} N - \lambda I)\bar{N} \bar{\zeta}, \dots, (\bar{N} N - \lambda I)^{m-1} \bar{N} \bar{\zeta} \quad (100)$$

are linearly independent. Suppose

$$\sum_{j=1}^m \alpha_j (\bar{N}N - \lambda I)^{j-1} \zeta + \sum_{j=1}^m \beta_j (\bar{N}N - \lambda I)^{j-1} \bar{N}\bar{\zeta} = 0 \quad (101)$$

where $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$ are complex numbers. Acting $(\bar{N}N - \lambda I)^{m-1}$ on both sides of (101), we get

$$(\bar{N}N - \lambda I)^{m-1}(\alpha_1 \zeta + \beta_1 \bar{N}\bar{\zeta}) = 0. \quad (102)$$

Then

$$\begin{aligned} & (|\alpha_1|^2 - \lambda|\beta_1|^2)(\bar{N}N - \lambda I)^{m-1} \zeta \\ &= -\bar{\alpha}_1 \beta_1 (\bar{N}N - \lambda I)^{m-1} \bar{N}\bar{\zeta} - \lambda|\beta_1|^2 (\bar{N}N - \lambda I)^{m-1} \zeta \\ &= -\bar{\alpha}_1 \beta_1 \bar{N} \overline{(\bar{N}N - \lambda I)^{m-1} \zeta} - \lambda|\beta_1|^2 (\bar{N}N - \lambda I)^{m-1} \zeta \\ &= \beta_1 \bar{N} \beta_1 \overline{(\bar{N}N - \lambda I)^{m-1} \bar{N}\bar{\zeta}} - \lambda|\beta_1|^2 (\bar{N}N - \lambda I)^{m-1} \zeta \\ &= |\beta_1|^2 \bar{N}N (\bar{N}N - \lambda I)^{m-1} \zeta - \lambda|\beta_1|^2 (\bar{N}N - \lambda I)^{m-1} \zeta \\ &= |\beta_1|^2 (\bar{N}N - \lambda I)^m \zeta = 0. \end{aligned} \quad (103)$$

Since $\lambda < 0$ and $(\bar{N}N - \lambda I)^{m-1} \zeta \neq 0$, we have $\alpha_1 = \beta_1 = 0$. Continuing this process by acting $(\bar{N}N - \lambda I)^{m-2}, \dots, (\bar{N}N - \lambda I)^0$ on both sides of (101) respectively, we get $\alpha_1 = \dots = \alpha_m = \beta_1 = \dots = \beta_m = 0$. This proves the linear independence of the vectors in (100). Let $\tilde{V}_1 = \text{span}\{\bar{N}\bar{\zeta}, (\bar{N}N - \lambda I)\bar{N}\bar{\zeta}, \dots, (\bar{N}N - \lambda I)^{m-1}\bar{N}\bar{\zeta}\}$. If $\bar{N}\bar{\zeta} = (\bar{N}N - \lambda I)\zeta' \in \text{Image}(\bar{N}N - \lambda I)$, then $\zeta = (\bar{N}N - \lambda I)N^{-1}\zeta' \in \text{Image}(\bar{N}N - \lambda I)$, which contradicts the choice of ζ . Hence $\bar{N}\bar{\zeta} \notin \text{Image}(\bar{N}N - \lambda I)$. Moreover, \tilde{V}_1 is invariant and irreducible under the action of $\bar{N}N - \lambda I$. Hence it must be one of V_j with $2 \leq j \leq m$, which means that m is even ($m = 2k$) and $\mathcal{R}_\lambda = (W_1 \oplus \widetilde{W}_1) \oplus \dots \oplus (W_k \oplus \widetilde{W}_k)$ where $(W_1, \dots, W_k, \widetilde{W}_1, \dots, \widetilde{W}_k)$ is a permutation of (V_1, \dots, V_{2k}) . Therefore, the multiplicity of each negative eigenvalue of $\bar{N}N$ must be even.

Thus, if the eigenvalues of $\bar{N}N$ are $\lambda_1, \dots, \lambda_{2n}$ (multiple eigenvalues are listed repeatedly), then

$$\begin{aligned} \det T &= |\det A|^2 \det(I + \bar{N}N) \\ &= |\det A|^2 \left(\prod_{\text{Re } \lambda_j \neq 0} |1 + \lambda_j|^2 \prod_{\lambda_j < 0} (1 + \lambda_j)^2 \right)^{1/2} \prod_{\lambda_j \geq 0} (1 + \lambda_j) \geq 0. \end{aligned} \quad (104)$$

If A or B is not invertible, it is a limit of invertible matrices, and the conclusion is also true. The lemma is proved.

Lemma 3 Suppose $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ where $\lambda_1, \dots, \lambda_n$ are distinct complex numbers such that $\bar{\lambda}_j \neq \pm i\lambda_l$ for all $j, l = 1, \dots, n$, $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$. Denote

$$V = \begin{pmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \dots & 1 \end{pmatrix}, \quad E = \text{diag}((i\varepsilon)^{n-1}, (i\varepsilon)^{n-2}, \dots, 1) \quad (105)$$

where $\varepsilon = \pm 1$. Let

$$T = \begin{pmatrix} V & \Gamma V E \\ -\bar{\Gamma} \bar{V} \bar{E} & \bar{V} \end{pmatrix}, \quad (106)$$

then there is a positive number C depending on Λ only, such that $\det T > C$.

Proof. Denote

$$S_k^{(\hat{m})} = \sum_{\substack{j_1 < \dots < j_k \\ j_1, \dots, j_k \neq m}} \lambda_{j_1} \cdots \lambda_{j_k}, \quad (107)$$

then

$$\sum_{k=1}^n (-1)^{k-1} S_{k-1}^{(\hat{m})} x^{n-k} = \prod_{\substack{s=1 \\ s \neq m}}^n (x - \lambda_s). \quad (108)$$

Since the entries of V are $V_{jk} = \lambda_j^{n-k}$, we have

$$(V^{-1})_{jk} = \frac{(-1)^{j-1} S_{j-1}^{(\hat{k})}}{\prod_{\substack{s=1 \\ s \neq k}}^n (\lambda_k - \lambda_s)}. \quad (109)$$

Hence

$$\begin{aligned} \det T &= \det(V) \det(\bar{V} + \bar{\Gamma} \bar{V} \bar{E} V^{-1} \Gamma V E) \\ &= |\det(V)|^2 \det(I + \bar{\Gamma} \bar{V} \bar{E} V^{-1} \Gamma V E \bar{V}^{-1}). \end{aligned} \quad (110)$$

For $j = 1, \dots, n$, let $\lambda_j = e^{i\pi/4} \delta r_j$ where $\delta = 1$ if $\varepsilon = 1$ and $\delta = i$ if $\varepsilon = -1$, then r_j 's are distinct and $\bar{r}_j \pm r_l \neq 0$ for all $j, l = 1, \dots, n$.

$$\begin{aligned} (\bar{V} \bar{E} V^{-1})_{jk} &= \sum_{l=1}^n (-i\varepsilon \bar{\lambda}_j)^{n-l} \frac{(-1)^{l-1} S_{l-1}^{(\hat{k})}}{\prod_{\substack{s=1 \\ s \neq k}}^n (\lambda_k - \lambda_s)} = \prod_{\substack{s=1 \\ s \neq k}}^n \frac{-i\varepsilon \bar{\lambda}_j - \lambda_s}{\lambda_k - \lambda_s} \\ &= (-1)^{n-1} \prod_{\substack{s=1 \\ s \neq k}}^n \frac{\bar{r}_j + r_s}{r_k - r_s} = (-1)^{n-1} (AMB^{-1})_{jk} \end{aligned} \quad (111)$$

where

$$A = (a_j \delta_{jk}), \quad B = (b_j \delta_{jk}), \quad M = (M_{jk}), \quad (112)$$

$$a_j = \prod_{s=1}^n (\bar{r}_j + r_s), \quad b_k = \prod_{\substack{s=1 \\ s \neq k}}^n (r_k - r_s), \quad M_{jk} = (\bar{r}_j + r_k)^{-1}. \quad (113)$$

Since $\overline{\bar{V} \bar{E} V^{-1}} \bar{V} \bar{E} V^{-1} = I$, we have $\overline{AMB^{-1}} = (AMB^{-1})^{-1}$. Hence

$$\begin{aligned} \det T &= |\det(V)|^2 \det(I + \bar{\Gamma} AMB^{-1} \Gamma BM^{-1} A^{-1}) \\ &= |\det(V)|^2 (\det M)^{-1} \det(M + \bar{\Gamma} M \Gamma) \end{aligned} \quad (114)$$

since Γ , A , B are diagonal matrices.

Suppose $\operatorname{Re}(r_1), \dots, \operatorname{Re}(r_m) < 0$, $\operatorname{Re}(r_{m+1}), \dots, \operatorname{Re}(r_n) > 0$ since $\bar{r}_j + r_j \neq 0$ for all j . Write

$$\begin{aligned} M &= \left(\frac{1}{\bar{r}_j + r_k} \right)_{n \times n} = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix}, \\ M + \bar{\Gamma} M \Gamma &= \left(\frac{1 + \bar{\gamma}_j \gamma_k}{\bar{r}_j + r_k} \right)_{n \times n} = \begin{pmatrix} N_{11} & N_{12} \\ N_{12}^* & N_{22} \end{pmatrix} \end{aligned} \quad (115)$$

where M_{11} and N_{11} , M_{12} and N_{12} , M_{22} and N_{22} are $m \times m$, $m \times (n-m)$, $(n-m) \times (n-m)$ matrices respectively. Then M_{11} and N_{11} are negative definite Hermitian matrices, so are M_{11}^{-1} and N_{11}^{-1} ; M_{22} and N_{22} are positive definite Hermitian matrices.

Let

$$\Gamma_1 = \operatorname{diag}(\gamma_1, \dots, \gamma_m), \quad \Gamma_2 = \operatorname{diag}(\gamma_{m+1}, \dots, \gamma_n), \quad (116)$$

then

$$\begin{aligned} \det(-N_{11}) &= \det(-M_{11} + \Gamma_1^*(-M_{11})\Gamma_1) \geq \det(-M_{11}), \\ \det(N_{22}) &= \det(M_{22} + \Gamma_2^*M_{22}\Gamma_2) \geq \det(M_{22}). \end{aligned} \quad (117)$$

Hence

$$\begin{aligned} (-1)^m \det(M + \bar{\Gamma} M \Gamma) &= \det(-N_{11}) \det(N_{22} + N_{12}^*(-N_{11})^{-1}N_{12}) \\ &\geq |\det M_{11}| \cdot |\det M_{22}|. \end{aligned} \quad (118)$$

On the other hand,

$$\begin{aligned} (-1)^m \det M &= \det(-M_{11}) \det(M_{22} + M_{12}^*(-M_{11})^{-1}M_{12}) \\ &\geq |\det M_{11}| \cdot |\det M_{22}| > 0, \end{aligned} \quad (119)$$

hence

$$\det T \geq |\det V|^2 \frac{|\det M_{11}| \cdot |\det M_{22}|}{|\det M|}. \quad (120)$$

The lemma is proved.

Lemma 4 Let $\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2)$ where Λ_1 and Λ_2 are $n \times n$ diagonal matrices satisfying $\bar{\Lambda}_2 = i\varepsilon\Lambda_1$ where $\varepsilon = \pm 1$. Let ζ be a $2n$ dimensional column vector. For $j \geq k$, denote

$$R_{j \dots k}^\Lambda = \begin{pmatrix} \Lambda^j \zeta & \Lambda^{j-1} \zeta, \dots, \Lambda^k \zeta \end{pmatrix}. \quad (121)$$

Let

$$\Pi^\Lambda = \begin{pmatrix} \Lambda^n \zeta & \Lambda^{n-1} \zeta & \Lambda^{n-2} \zeta & R_{n-3 \dots 0}^\Lambda & K_n \bar{R}_{n-1 \dots 0}^\Lambda & 0 & 0 \\ \Lambda^{n+1} \zeta & \Lambda^n \zeta & \Lambda^{n-1} \zeta & 0 & 0 & R_{n-2 \dots 0}^\Lambda & K_n \bar{R}_{n-1 \dots 0}^\Lambda \end{pmatrix}, \quad (122)$$

where $K_n = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$, then $\det \Pi^\Lambda$ is purely imaginary.

Proof. $\bar{\Lambda}_2 = i\varepsilon\Lambda_1$ is equivalent to $\Lambda K_n = -i\varepsilon K_n \bar{\Lambda}$. Denote $d^\Lambda = \det \Delta^\Lambda$. Multiplying row j of d^Λ by $-\lambda_j$ and adding it to row $2n+j$ ($j = 1, \dots, 2n$), we get

$$d^\Lambda = \begin{vmatrix} \Lambda^n \zeta & \Lambda^{n-1} \zeta & \Lambda^{n-2} \zeta & R_{n-3\dots 0}^\Lambda & K_n \bar{R}_{n-1\dots 0}^\Lambda & 0 & 0 \\ 0 & 0 & 0 & -R_{n-2\dots 1}^\Lambda & i\varepsilon K_n \bar{R}_{n\dots 0}^\Lambda & R_{n-2\dots 0}^\Lambda & K_n \bar{R}_{n-1\dots 0}^\Lambda \end{vmatrix} \quad (123)$$

by using $\Lambda K_n = -i\varepsilon K_n \bar{\Lambda}$. Adding columns $2n+2, \dots, 3n-1$ to columns $4, \dots, n+1$, multiplying columns $3n+1, \dots, 4n-1$ by $-i\varepsilon$ and adding them to columns $n+3, \dots, 2n+1$, we get

$$d^\Lambda = \begin{vmatrix} R_{n\dots 0}^\Lambda & K_n \bar{\Lambda}^{n-1} \bar{\zeta} & K_n \bar{R}_{n-2\dots 0}^\Lambda & 0 & 0 \\ 0 & i\varepsilon K_n \bar{\Lambda}^n \bar{\zeta} & 0 & R_{n-2\dots 0}^\Lambda & K_n \bar{R}_{n-1\dots 0}^\Lambda \end{vmatrix}, \quad (124)$$

By moving the columns,

$$\begin{aligned} d^\Lambda &= \begin{vmatrix} R_{n\dots 0}^\Lambda & K_n \bar{R}_{n-2\dots 0}^\Lambda & 0 & K_n \bar{\Lambda}^{n-1} \bar{\zeta} & 0 \\ 0 & 0 & R_{n-2\dots 0}^\Lambda & i\varepsilon K_n \bar{\Lambda}^n \bar{\zeta} & K_n \bar{R}_{n-1\dots 0}^\Lambda \end{vmatrix} \\ &= i\varepsilon \begin{vmatrix} R_{n\dots 0}^\Lambda & K_n \bar{R}_{n-2\dots 0}^\Lambda \end{vmatrix} \cdot \begin{vmatrix} R_{n-2\dots 0}^\Lambda & K_n \bar{R}_{n\dots 0}^\Lambda \end{vmatrix}, \end{aligned} \quad (125)$$

which is purely imaginary according to Lemma 1. The lemma is proved.

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